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FORCED VIBRATION RESPONSES OF A VISCOELASTIC STRUCTURE

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1. INTRODUCTION

In the theory of linear viscoelasticity, one of the hereditary models [1–3] is a constitutive law of the form (a one-dimensional element is taken as an illustration)

$$\sigma(t) = E\left(e(t) - \int_0^t \Gamma(t-\tau)e(\tau)\,\mathrm{d}\tau\right),$$

which can be written in operator form as

$$\sigma = \tilde{E}e,$$

and

$$\tilde{E}(\,\cdot\,) = E\left[1(\,\cdot\,) - \int_0^t \Gamma(t-\tau)(\,\cdot\,)\,\mathrm{d}\tau\right]. \tag{1}$$

The scalar function $\Gamma(t-\tau)$ is called the relaxation kernel and E is the instantaneous (or elastic) Young's modulus. The quantity $\tilde{E}$ will be called Young modulus operator. One can introduce a similar representation of the Poisson ratio operator [2]:

$$\tilde{\nu}(\,\cdot\,) = \nu\left[1(\,\cdot\,) + \int_0^t \Lambda(t-\tau)(\,\cdot\,)\,\mathrm{d}\tau\right], \tag{2}$$

where the quantity ν is called instantaneous (or elastic) Poisson's ratio.

Application of the finite element method to elastic systems allows the formulation of dynamic problems in terms of mass and stiffness matrices; vectors of displacement (response) and force (excitation). For viscoelastic systems, E (Young's modulus) and ν (Poisson's ratio) should be replaced by their hereditary analogs (operators). In the case of the finite element method, this implies the replacement of the material constants E, ν in the stiffness matrix by their viscoelastic analogs (operators) $\tilde{E}$, $\tilde{\nu}$.

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In the present paper, a viscoelastic structure is assumed to consist of isotropic homogeneous hereditary materials and the materials are assumed to be in an isothermal state. The principal equations to be used are: (1) the constitutive relation between stress and strain of the material and (2) the equation of motion of the structure.

Finite element method applications for dynamic viscoelastic systems are usually described in the literature in the context of time numerical integration schemes, see for example [4, 5], or are based on numerical inversion algorithms of the Laplace transformed solution [6, 7]. Descriptions of numerical methods for Volterra's equations can be found, e.g., in references [8, 9].

The authors of reference [6] employed a fractional constitutive model and formulated a method for decoupling the equations of motion of a multi-degree-of-freedom-system in the Laplace domain. However, in the constitutive relation just one parameter—shear modulus—was introduced (instead of two for an isotropic medium), which reduces the universality of the method. Also, the inverse Laplace transformation of the obtained solution required significant computational effort.

In one of the widely used models of the constitutive viscoelastic law, the relaxation kernel is represented by a sum of exponentials (Prony series). This model is simpler than fractional ones and, as will be shown in the present paper, assumes analytical homogeneous solutions and, in some cases, analytical forced responses.

The authors of reference [7] employed a constitutive model with a relaxation kernel represented by exponentials; however, the model assumed some restrictions imposed on the coefficients in the exponential series. The Laplace transform approach was also employed.

A construction of homogeneous analytical solutions to the free vibration problem has been shown in reference [10] using the Laplace transform method, where the relaxation kernel was represented by a Prony series without any restrictions imposed on the coefficients. In references [11, 12], a different approach (named the "substitution method") was developed to determine the unknown parameters which are involved in the analytical homogeneous solution. The case of periodic forced excitation was also covered there.

Homogeneous solutions will be used in this study to construct the forced vibration response of a structure to an arbitrary excitation. The forced response representation, using a convolution integral, is well known and used for linear damped systems. However, there appear to be no reports in the literature which show use of the convolution integral for the case of a viscoelastic structure, the equation of motion of which is described by a Volterra (integro-differential) type equation.

The approach developed in this study allows the forced vibration solution to be obtained without numerical integration of the equation of motion and without use of Laplace transforms. In cases when the convolution integral can be evaluated analytically (it depends on the type of forcing function), the general solution is completely in analytical form. In other cases, a numerical evaluation of the integral is required, which would take less computation and provide better accuracy than the integration of the whole equation of motion.

2. SOLUTION PROCEDURE

One can begin with the case when the Poisson ratio operator $\tilde{v}$ in equation (2) is elastic, i.e., constant, $\tilde{v} = v$ and a viscoelastic structure consists of one homogeneous material.

Denote by M the mass matrix and by K the elastic stiffness matrix of the structure. For a homogeneous structure, one can represent the stiffness matrix as

$$K = EK_0, \tag{3}$$

where E is the instantaneous Young's modulus. For a viscoelastic structure, it is necessary to replace E by the viscoelastic operator $\tilde{E}$ (see equation (1)). The relaxation kernel in equation (1) will be assumed as a sum of exponentials:

$$\Gamma(t-\tau) = \sum_{i=1}^{n} a_i \, \mathrm{e}^{-\alpha_i(t-\tau)}. \tag{4}$$

The equation of forced motion can be written as

$$M\ddot{X} + C\dot{X} + \tilde{E}K_0X = F, \tag{5}$$

where M, C, and K_0 are $m \times m$ constant matrices, X is a vector function of displacements and F is a vector of forcing functions. The term $C\dot{X}$ takes into account the presence of viscous damping.

Representing equation (5) in state-space form, one obtains

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}\begin{bmatrix} \dot{X} \\ \ddot{X} \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}\begin{bmatrix} X \\ \dot{X} \end{bmatrix} - \int_0^t \Gamma(t-\tau)\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X(\tau) \\ \dot{X}(\tau) \end{bmatrix} \mathrm{d}\tau = \begin{bmatrix} F \\ 0 \end{bmatrix}, \tag{6}$$

where K was defined by equation (3). Introducing notations

$$Y = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}, \qquad D = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \qquad A = -\begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix};$$

$$B(t-\tau) = \Gamma(t-\tau)\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{F} = \begin{bmatrix} F \\ 0 \end{bmatrix},$$

equation (6) can be rewritten as

$$D\dot{Y} = AY + \int_0^t B(t-\tau)Y(\tau)\,\mathrm{d}\tau + \tilde{F}$$

and, premultiplying by D^{-1} one can write

$$\dot{Y} = GY + \int_0^t H(t-\tau)Y(\tau)\,\mathrm{d}\tau + R, \tag{7}$$

where $G = D^{-1}A$, $H(t-\tau) = D^{-1}B(t-\tau)$ and $R = D^{-1}\tilde{F}$. Note that

$$D^{-1} = \begin{bmatrix} 0 & M^{-1} \\ M^{-1} & -M^{-1}CM^{-1} \end{bmatrix},$$

therefore,

$$R = \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}. \tag{8}$$

Note that the upper portion of equation (6) repeats equation (5) and the lower portion is satisfied identically. Therefore, equations (5) and (7) are equivalent and their solutions (homogeneous and general ones) are essentially the same (they differ only by the form of the representation).

Now, consider a homogeneous version of equation (7):

$$\dot{Z} = GZ + \int_0^t H(t-\tau)Z(\tau)\,\mathrm{d}\tau. \tag{9}$$

Introduce the principal matrix solution $Z(t)$, the columns of which are homogeneous solutions corresponding to the unit initial conditions, i.e., $Z(0) = I$. According to a theorem from reference [13], such a matrix exists and is unique. The columns of $Z(t)$ are linear independent vector functions.

Using another theorem from reference [13], one can represent the solution of equation (7) in the form

$$Y(t) = Z(t)\tilde{X}(0) + \int_0^t Z(t-\tau)R(\tau)\,\mathrm{d}\tau, \tag{10}$$

where the initial vector

$$\tilde{X}(0) = \begin{bmatrix} X_0 \\ \dot{X}_0 \end{bmatrix}. \tag{11}$$

The only assumption implied in this theorem is that the components of R and H are continuous or piece-wise continuous functions in $[0, \infty)$.

Note also that, due to equation (8), only columns of $Z(t)$ associated with the initial velocities are needed in the integral term of equation (10).

In this study, the columns of $Z(t)$ (homogeneous solutions) will be represented and used in analytical form. A practical method of their derivation was developed in references [11, 12] and the principal results from these sources are utilized here.

The ith column of matrix $Z(t)$ consists of a subvector Z^d of displacements and a subvector Z^v of velocities, i.e.,

$$Z_i = \begin{bmatrix} Z_i^d \\ Z_i^v \end{bmatrix}. \tag{12}$$

The ith independent solution can be represented (see references [11, 12]) as

$$Z_i^d(t) = \sum_{j=1}^{m(n+2)} c_j X_j \, e^{p_j t}, \qquad Z_i^v(t) = \dot{Z}_i^d(t) = \sum_{j=1}^{m(n+2)} c_j X_j \, p_j \, e^{p_j t}. \tag{13}$$

Here, X_j is a complex vector ($m \times 1$), c_j, p_j are complex scalars, m is the number of degrees of freedom (size of matrices, M, C, K), and n is the number of terms in the relaxation kernel. Note that in equation (12), Z^d constitutes a solution to the homogeneous case of equation (5). The complex coefficients c_j (different for each ith initial condition) are determined from the system (16) with corresponding initial conditions, i.e., in such a way that the ith component in $\tilde{X}$ (see equation (11)) equals 1 with all the rest being 0.

The values of p_j and X_j are determined as a result of a solution of an eigenvalue problem. This eigenvalue problem was formulated (see details in references [11, 12]) as

$$\left(p \begin{bmatrix} B_1 & B_2 & B_3 & \cdots & B_{n+2} \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} + \begin{bmatrix} B_0 & 0 & 0 & \cdots & 0 \\ 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I \end{bmatrix} \right) \hat{Q} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{14}$$

or in abbreviated form,

$$(p\hat{A} + \hat{B})\hat{Q} = 0,$$

where the jth eigenvector will be

$$\hat{Q}_j = \begin{bmatrix} X_j \\ p_j X_j \\ p_j^2 X_j \\ \vdots \\ p_j^{n+1} X_j \end{bmatrix}, \quad j = 1, m(n+2).$$

The matrix coefficient $B_{n+2} = M$ and other coefficients can be readily evaluated. For example, for the Kelvin–Voigt model ($n = 1$ in equation (4)), i.e., for the relaxation kernel $\Gamma(t-\tau) = a_1 \, e^{-\alpha_1(t-\tau)}$, these matrix coefficients are

$$B_0 = K(\alpha_1 - a_1), \qquad B_1 = C\alpha_1 + K, \qquad B_2 = M\alpha_1 + C, \qquad B_3 = M. \quad (15)$$

For the case $n = 2$, i.e., for the relaxation kernel $\Gamma(t-\tau) = a_1 \, e^{-\alpha_1(t-\tau)} + a_2 \, e^{-\alpha_2(t-\tau)}$:

$$B_0 = (\alpha_1\alpha_2 - a_1\alpha_2 - a_2\alpha_1)K, \qquad B_1 = \alpha_1\alpha_2 C + (\alpha_1 + \alpha_2 - a_1 - a_2)K,$$

$$B_2 = \alpha_1\alpha_2 M + K + (\alpha_1 + \alpha_2)C, \qquad B_3 = (\alpha_1 + \alpha_2)M + C, \qquad B_4 = M.$$

For the case $n = 3$:

$$B_0 = (\alpha_1\alpha_2\alpha_3 - a_1\alpha_2\alpha_3 - a_2\alpha_1\alpha_3 - a_3\alpha_1\alpha_2)K,$$

$$B_1 = \alpha_1\alpha_2\alpha_3 C + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 - a_1(\alpha_2 + \alpha_3) - a_2(\alpha_1 + \alpha_3) - a_3(\alpha_1 + \alpha_2))K,$$

$$B_2 = \alpha_1\alpha_2\alpha_3 M + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)C - (a_1 + a_2 + a_3)K,$$

$$B_3 = (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)M + (\alpha_1 + \alpha_2 + \alpha_3)C + K,$$

$$B_4 = (\alpha_1 + \alpha_2 + \alpha_3)M + C, \qquad B_5 = M.$$

A system of linear $m(n+2)$ equations with respect to $m(n+2)$ unknowns c_j was formulated in references [11, 12]. In matrix form, this system can be written as

$$\begin{bmatrix} \frac{a_1}{p_1 + \alpha_1} X_1 & \frac{a_1}{p_2 + \alpha_1} X_2 & \cdots & \frac{a_1}{p_{m(n+2)} + \alpha_1} X_{m(n+2)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{a_n}{p_1 + \alpha_n} X_1 & \frac{a_n}{p_2 + \alpha_n} X_2 & \cdots & \frac{a_n}{p_{m(n+2)} + \alpha_n} X_{m(n+2)} \\ X_1 & X_2 & \cdots & X_{m(n+2)} \\ p_1 X_1 & p_2 X_2 & \cdots & p_{m(n+2)} X_{m(n+2)} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_{m(n+2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \cdots \\ 0 \\ X_0 \\ \dot{X}_0 \end{bmatrix} \quad (16)$$

The basic part X_j of the eigenvector $\hat{Q}_j$ is used in equations (16) and (13).

Remark 1. Note a principal difference between the solution of equation (9) and the solution of the equation

$$\dot{Y} = GY, \quad (17)$$

where G is a constant $2m \times 2m$ matrix. For the latter equation, assuming the existence of $2m$ linear independent eigenvectors, the solution can be expressed as

$$Y(t) = \sum_{j=1}^{2m} d_j Y_j \, e^{\xi_j t}, \tag{18}$$

where the coefficients d_j are arbitrary and ξ_j are, in general, complex eigenvalues. Each exponential term $d_j Y_j \, e^{\xi_j t}$ in equation (18) is a solution of equation (17). However, in the case of equation (9), the coefficients c_j, $j = 1, m(n+2)$ in the solution

$$Z^d(t) = \sum_{j=1}^{m(n+2)} c_j X_j \, e^{p_j t} \tag{19}$$

should satisfy system (16). It is clear that if, for instance, $c_k \neq 0$ and all the rest $c_j = 0$ ($j = 1, m(n+2), j \neq k$), then the system (16) may not be satisfied regardless of the initial conditions. Namely, the upper portion (where the right side has 0 components) may be violated.

In other words, the solution of equation (9) in the form (19) does not imply that, in general, each exponential term of equation (19) is separately a solution of equation (9). Note also that each independent solution Z_i (ith column of matrix $Z(t)$) may contain, in general, several exponential terms.

Remark 2. When the forcing function $F(t)$ is such that the convolution integral in equation (10) allows for analytical evaluation, then the whole solution (10) will be analytical. In other cases, numerical evaluation of the integral is required.

Before proceeding with the presentation of numerical results, the method for obtaining homogeneous solutions is extended to the case when Poisson's ratio is considered as a viscoelastic operator and the structure may consist of different materials.

2.1. *Homogeneous solutions when Poisson's ratio is not constant*

The finite element method yields a mass matrix of the system M and an elastic stiffness matrix K. The derivation of the system stiffness matrix starts with an element stiffness matrix, where the constitutive law is involved. For a linear elastic material and 3-D stress–strain state:

$$[\sigma] = [E][\epsilon],$$

where

$$[\sigma] = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{13} \quad \sigma_{23}]^{\mathrm{T}}, \qquad [\epsilon] = [\epsilon_{11} \quad \epsilon_{22} \quad \epsilon_{33} \quad \epsilon_{12} \quad \epsilon_{13} \quad \epsilon_{23}]^{\mathrm{T}},$$

$$[E] = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2G & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2G & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{bmatrix}.$$

where

$$\lambda = vE/((1+v)(1-2v)), \qquad 2G = E/(1+v). \tag{20}$$

One notices that matrix $[E]$ assumes the form

$$[E] = \lambda I_1 + 2GI, \quad \text{with} \quad I_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and I a unit matrix. Therefore, the stiffness matrix of a structure consisting of one homogeneous material can be represented as

$$K = \lambda K_1 + 2GK_2,$$

The hereditary analog of this matrix is obtained by replacement of E, v by corresponding operators, i.e., $E \to \tilde{E}$, $v \to \tilde{v}$, or by another replacement $\lambda \to \tilde{\lambda}$, $G \to \tilde{G}$.

Viscoelastic Lamé operators $\tilde{\lambda}$ and $\tilde{G}$ are assumed in the form

$$\tilde{\lambda} y(t) = \lambda \left(y(t) - \int_0^t \sum_{i=1}^{n} a_i \, \mathrm{e}^{-\alpha_i (t-\tau)} y(\tau) \, \mathrm{d}\tau \right),$$

$$\tilde{G} y(t) = G \left(y(t) - \int_0^t \sum_{i=1}^{n} b_i \, \mathrm{e}^{-\gamma_i (t-\tau)} y(\tau) \, \mathrm{d}\tau \right). \tag{21, 22}$$

If the expressions for operators $\tilde{E}$ and $\tilde{v}$ are known, then the Lamé operators can be calculated by using operator multiplication and division rules [2], and formulas (20).

Now, the equation of free motion can be written in the form

$$M\ddot{X} + C\dot{X} + (\tilde{\lambda} K_1 + 2\tilde{G} K_2) X = 0. \tag{23}$$

The solution of equation (23) is sought in the form

$$X(t) = \sum_{j=1}^{m(2n+2)} c_j X_j \, \mathrm{e}^{p_j t}, \tag{24}$$

where X_j is a complex vector $(m \times 1)$; c_j, p_j are complex, m is the number of degrees of freedom, and n is the number of exponential terms in equations (21) and (22). Note that the number of exponential terms in solution (24) is taken for this case as $m(2n+2)$ in comparison with $m(n+2)$ in equations (13) when only Young's modulus was considered as a viscoelastic operator.

Denoting $N = m(2n + 2)$ and substituting equation (24) into equation (23), using equations (21) and (22), one obtains

$$\sum_{j=1}^{N}\left[p_j^2 M + p_j C + \lambda K_1\left(1 - \sum_{i=1}^{n}\frac{a_i}{p_j + \alpha_i}\right) + 2GK_2\left(1 - \sum_{i=1}^{n}\frac{b_i}{p_j + \gamma_i}\right)\right]c_j X_j\, e^{p_j t}$$

$$+ \sum_{i=1}^{n}\left[\lambda K_1 \sum_{j=1}^{N}\frac{a_i}{p_j + \alpha_i}\, c_j X_j\right]\mathrm{e}^{-\alpha_i t} + \sum_{i=1}^{n}\left[2GK_2 \sum_{j=1}^{N}\frac{b_i}{p_j + \gamma_i}\, c_j X_j\right] e^{-\gamma_i t} = 0.$$

Therefore the following equations need to be satisfied.

$$\left[p_j^2 M + p_j C + \lambda K_1\left(1 - \sum_{i=1}^{n}\frac{a_i}{p_j + \alpha_i}\right) + 2GK_2\left(1 - \sum_{i=1}^{n}\frac{b_i}{p_j + \gamma_i}\right)\right]X_j = 0, \quad j = 1, N. \tag{25}$$

Relations (25) generate an eigenvalue problem analogous to equation (14).

Also, the following terms should be set to zero:

$$\sum_{j=1}^{N}\frac{a_i}{p_j + \alpha_i}\, c_j X_j = 0, \qquad i = 1, n \tag{26}$$

and

$$\sum_{j=1}^{N}\frac{b_i}{p_j + \gamma_i}\, c_j X_j = 0, \qquad i = 1, n. \tag{27}$$

Relations (26) and (27) and the initial conditions constitute a system of $N = m(2n + 2)$ linear equations with respect to N unknowns c_j analogous to equation (16):

$$\begin{bmatrix}
\frac{a_1}{p_1 + \alpha_1} X_1 & \frac{a_1}{p_2 + \alpha_1} X_2 & \cdots & \frac{a_1}{p_N + \alpha_1} X_N \\
\frac{b_1}{p_1 + \gamma_1} X_1 & \frac{b_1}{p_2 + \gamma_1} X_2 & \cdots & \frac{b_1}{p_N + \gamma_1} X_N \\
\vdots & \vdots & \vdots & \vdots \\
\frac{a_n}{p_1 + \alpha_n} X_1 & \frac{a_n}{p_2 + \alpha_n} X_2 & \cdots & \frac{a_n}{p_N + \alpha_n} X_N \\
\frac{b_n}{p_1 + \gamma_n} X_1 & \frac{b_n}{p_2 + \gamma_n} X_2 & \cdots & \frac{b_n}{p_N + \gamma_n} X_N \\
X_1 & X_2 & \cdots & X_N \\
p_1 X_1 & p_2 X_2 & \cdots & p_N X_N
\end{bmatrix}
\begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix}
=
\begin{bmatrix} 0 \\ \cdots \\ 0 \\ X_0 \\ \dot{X}_0 \end{bmatrix}.$$

Having found c_j, $j = 1$, $m(2n + 2)$, the homogeneous solution (24) is found.

Note that extension of this method to the case of inhomogeneous structures is accomplished in a straightforward way, i.e., by representing the system stiffness matrix operator as

$$\tilde{K} = \sum_{j=1}^{L} (\tilde{\lambda}_j K_{1j} + 2\tilde{G}_j K_{2j}),$$

where L is the number of homogeneous substructures. Then, the solution is sought in the form

$$X(t) = \sum_{j=1}^{m(2L \times n + 2)} c_j X_j \, e^{p_j t}, \tag{28}$$

where n is the number of exponentials in the relaxation kernels of operators $\tilde{\lambda}_j$, $\tilde{G}_j$; m is the number of degrees of freedom (size of matrices M, C, K_{1j}, K_{2j}, $j = 1, L$). Note that the number of exponential terms in solution (28) is taken for this case as $m(2L \times n + 2)$ in comparison with $m(2n + 2)$ in equation (24) when one homogeneous component constituted the whole system. The formulation of an eigenvalue problem analogous to equation (14) and a system analogous to equation (16) then follow. Thus, an analytical homogeneous solution in the form (28) can be found for the case of inhomogeneous structures and when two parameters in the constitutive law are considered as operators: Young's modulus and Poisson's ratio, or two Lamé operators. This solution can then be used in equation (10) to construct the general solution.

Remark. In the case of an inhomogeneous structure consisting of L components with only Young's moduli introduced as operators, the homogeneous solution is sought in the form

$$X(t) = \sum_{j=1}^{m(L \times n + 2)} c_j X_j \, e^{p_j t}. \tag{29}$$

Note that the number of exponential terms in the solution (29) is taken for this case as $m(L \times n + 2)$ in comparison with $m(2L \times n + 2)$ in equation (28) when two Lamé operators were introduced.

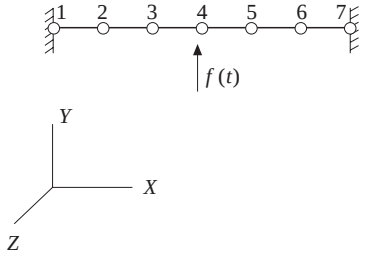

Figure 1. Viscoelastic beam with fixed ends.

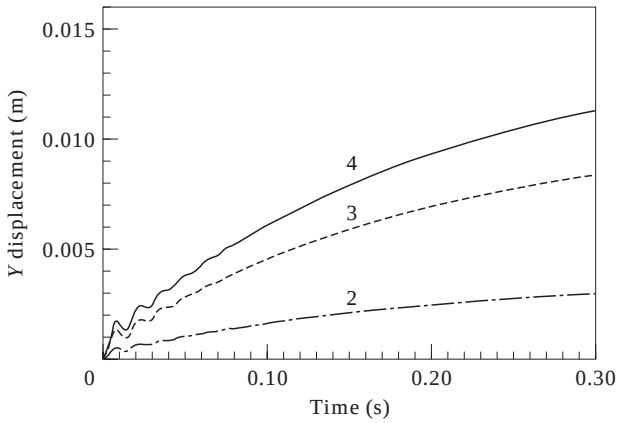

Figure 2. Forced vibration response to step-loading with $a_1 = 95\ s^{-1}$; $\alpha_1 = 100\ s^{-1}$. Key for Y-displacements: –·–·, node 2; -----, node 3; ——, node 4.

3. EXAMPLES OF COMPUTED FORCED RESPONSES

A program was written which calculates analytical homogeneous and forced vibration solutions according to the above described procedure.

The purpose of this section is to show practical feasibility of the procedure and to provide an example (with all the necessary input data) which can be used for comparison purposes by other researchers.

Numerical results are presented for a viscoelastic homogeneous beam with fixed ends (Figure 1). The parameters of the beam's cross section were $0{\cdot}01 \times 0{\cdot}01$ m, the beam's length was $0{\cdot}12$ m, the instantaneous Young's modulus E was $0{\cdot}15\text{E} + 08$ Pa, Poisson's ratio was assumed constant as $v = 0{\cdot}3$, and the density of the material was $0{\cdot}141\text{E} + 04\ \text{kg/m}^3$. The beam was meshed by 6 general (Timoshenko's type) beam elements. Each node of a beam element had 6 degrees of freedom (3 linear and 3 rotational). Thus, the size of the problem (number of unconstrained degrees of freedom) was $m = 30$.

The relaxation kernel in equation (4) was assumed as

$$\Gamma(t - \tau) = a_1\, \mathrm{e}^{-\alpha_1(t-\tau)}, \tag{30}$$

which corresponds to the Kelvin–Voigt model. The matrix coefficients B_i in equation (14) for this case ($n = 1$) were computed by equation (15). To investigate the effect of the hereditary part in equation (5), and not its combined effect with the viscous damping term $C\dot{X}$, it was assumed that the damping matrix $\mathbf{C} = 0$.

The size of the eigenvalue problem (14) and of the system (16) was $m(n + 2) = 30*(1 + 2) = 90$. Subroutine "DREIGN" [14] was used for eigenproblem (14) and subroutine "CDSOLN" for equation (16).

A vertical force $f(t)$ was applied at node 4 (Figure 1). At first, a step-type loading was considered (units N):

$$f(t) = \begin{Bmatrix} 0, & t < 0 \\ 1, & t \subset [0, \infty) \end{Bmatrix}.$$

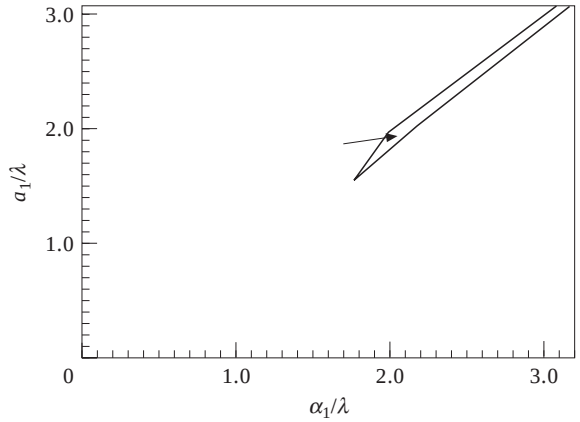

Figure 3. Overdamped region.

Note that for this case the convolution integral in equation (10) allows for an analytical evaluation.

In the first example, the values of the parameters in equation (30) were $a_1 = 95\ s^{-1}$ and $\alpha_1 = 100\ s^{-1}$. The ratio of the long-time modulus E_∞ to the instantaneous modulus E for this case was [2] $E_\infty/E = 1 - a_1/\alpha_1 = 0{\cdot}05$.

The numerical results in terms of the displacements are presented in Figure 2, where the graphs with numbers 2, 3 and 4 correspond to Y displacements of nodes 2, 3 and 4, respectively. All degrees of freedom had zero initial conditions. One can see the oscillatory character of the response at the initial stage. The motion was created mostly by the 1st free vibration mode.

The first natural frequency of the undamped beam in Figure 1 (if the hereditary term was set to zero) was 73·38 Hz, of $\lambda = 461{\cdot}09\ s^{-1}$. Using the results from reference [15], where a single-degree-of-freedom (SDOF) system was analyzed, one can provide an overdamped response if the point $(\alpha_1/\lambda;\ a_1/\lambda)$ is put into the

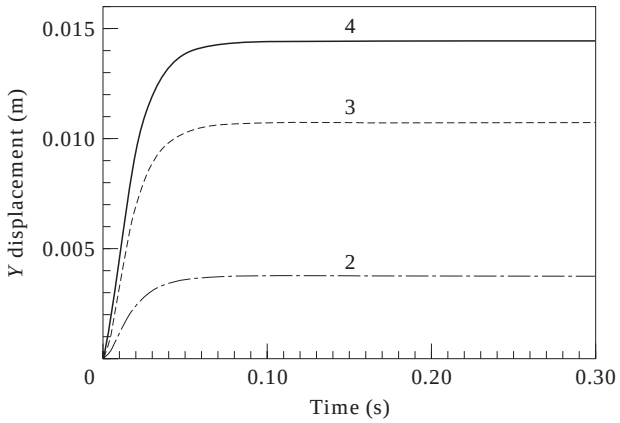

Figure 4. Forced vibration response to step-loading with $a_1 = 950\ s^{-1}$; $\alpha_1 = 1000\ s^{-1}$. Key as for Figure 2.

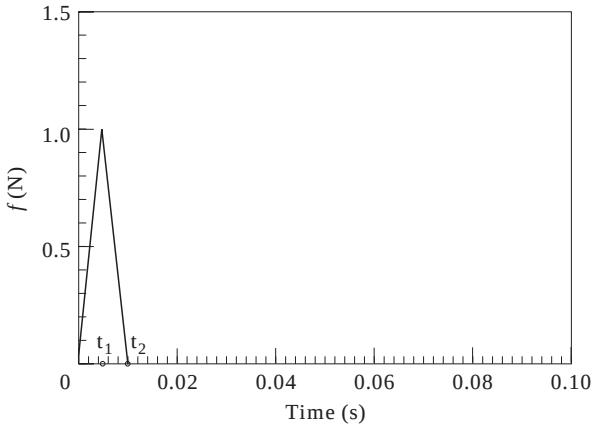

Figure 5. Forcing function: Δ-shape loading.

overdamped region. This region was built in reference [15] and is shown in Figure 3 by the arrow. Assuming that the forced response was mostly created by the first mode (for our example), one can use analogy with a SDOF system. Thus, one can prescribe values for α_1 and a_1 in such a way that a non-oscillatory (overdamped) character is achieved. Namely, the following values were prescribed: $a_1 = 950\ \mathrm{s}^{-1}$ and $\alpha_1 = 1000\ \mathrm{s}^{-1}$. Note that the point ($\alpha_1/\lambda = 1000/461{\cdot}09 = 2{\cdot}169$; $a_1/\lambda = 950/461{\cdot}09 = 2{\cdot}06$) belongs to the overdamped region (Figure 3). As expected, the response for this case was non-oscillatory. It is shown in Figure 4.

As next example, a Δ-shape forcing function $f(t)$ (Figure 5) was considered. Note that, for this type of loading, the convolution integral in equation (10) also allows for analytical evaluation, namely, one can subdivide the interval of integration in the convolution terms in three subintervals: $[0, t_1]$, $[t_1, t_2]$ and $[t_2, \infty)$. Then, the following formulae are useful for the evaluation of the convolution term:

$$\int_0^t \mathrm{e}^{p(t-\tau)} f(\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{e}^{p(t-\tau)} \gamma\tau\,\mathrm{d}\tau = \gamma\,\mathrm{e}^{pt} \int_0^t \mathrm{e}^{-p\tau}\tau\,\mathrm{d}\tau$$

$$= \frac{-\gamma(pt + 1 - \mathrm{e}^{pt})}{p^2}, \quad t \subset [0, t_1];$$

$$\int_0^t \mathrm{e}^{p(t-\tau)} f(\tau)\,\mathrm{d}\tau = \gamma\,\mathrm{e}^{pt} \int_0^{t_1} \mathrm{e}^{-p\tau}\tau\,\mathrm{d}\tau + \mathrm{e}^{pt} \int_{t_1}^t \mathrm{e}^{-p\tau}(\beta\tau + r)\,\mathrm{d}\tau, \quad t \subset [t_1, t_2];$$

$$\int_0^t \mathrm{e}^{p(t-\tau)} f(\tau)\,\mathrm{d}\tau = \gamma\,\mathrm{e}^{pt} \int_0^{t_1} \mathrm{e}^{-p\tau}\tau\,\mathrm{d}\tau + \mathrm{e}^{pt} \int_{t_1}^{t_2} \mathrm{e}^{-p\tau}(\beta\tau + r)\,\mathrm{d}\tau, \quad t \subset [t_2, \infty];$$

where γ, β, and r are constant coefficients. Recall that $f(t) = 0$ for $t > t_2$.

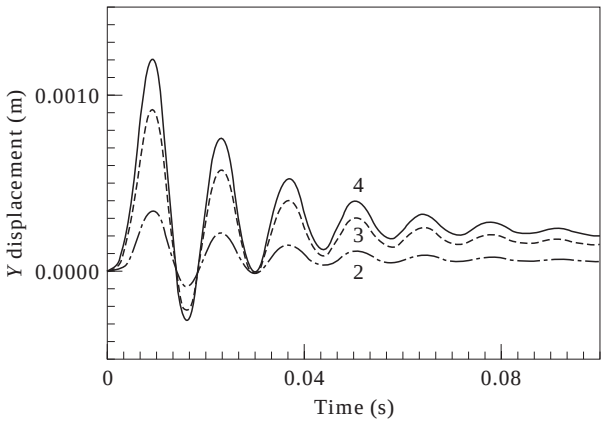

Figure 6. Forced vibration response to Δ-shape loading with $a_1 = 95\ s^{-1}$; $\alpha_1 = 100\ s^{-1}$. Key as for Figure 2.

The forced vibration response for this case of loading is presented in Figure 6 for $a_1 = 95\ s^{-1}$, $\alpha_1 = 100\ s^{-1}$. All degrees of freedom had zero initial conditions. One can see the oscillatory character of the response. Again the motion is created mostly by the first free vibration mode.

The forced vibration response for $a_1 = 950\ s^{-1}$, $\alpha_1 = 1000\ s^{-1}$ is shown in Figure 7. The response for this case was overdamped.

The last example was produced for values $a_1 = 750\ s^{-1}$ and $\alpha_1 = 1000\ s^{-1}$. It is shown in Figure 8. Again, one can see the oscillatory character of the response. Note that, in this case, the ratio a_1/λ was 1·626 and the point $(\alpha_1/\lambda; a_1/\lambda) = (2{\cdot}169, 1{\cdot}626)$ was outside of the overdamped region.

4. SUMMARY

A procedure for obtaining the forced vibration response to an arbitrary forced excitation for discrete viscoelastic structures has been shown for the case when the

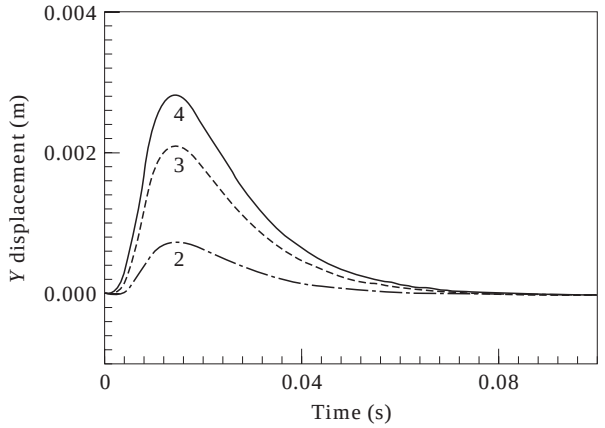

Figure 7. Forced vibration response to Δ-shape loading with $a_1 = 950\ s^{-1}$; $\alpha_1 = 1000\ s^{-1}$. Key as for Figure 2..

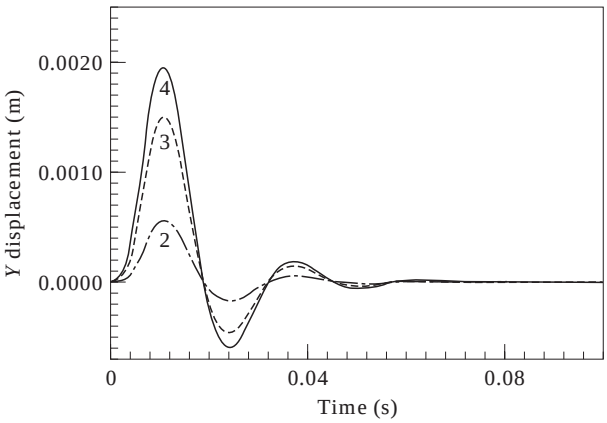

Figure 8. Forced vibration response to Δ-shape loading with $a_1 = 750\ s^{-1}$; $\alpha_1 = 1000\ s^{-1}$. Key as for Figure 2.

relaxation kernels in the constitutive relation are represented as a series of exponentials.

The procedure employs a convolution integral which yields the particular solution to the problem. The homogeneous term has been obtained in analytical form for the case of an inhomogeneous structure with Poisson's ratio and Young's modulus considered as viscoelastic operators.

The general solution is obtained in the time domain without the necessity to apply Laplace transforms or numerical integration for the equation of motion. For some types of the forcing function, an analytical expression for the convolution integral can be obtained.

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